12 Vectors and the Geometry of Space

12.1 Three-dimension coordinate Systems

1. coordinate axes, coordinate planes, octants, projections
2. distance formula between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

   $$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

   as the length of the diagonal of the box that has P_1 and P_2 on opposite corners
3. equation of a sphere with center $C(h, k, \ell)$ and radius r is

   $$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2$$

12.2 Vectors

1. vectors
2. sum/difference of vectors and scalar multiplication is component wise
3. given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector

   $$\vec{AB} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 >$$

4. the magnitude of the vector $< a_1, a_2, a_3 >$ is $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
5. properties of vectors
6. standard basis vectors

12.3 Dot product

1. if $a =< a_1, a_2, a_3 >$ and $b =< b_1, b_2, b_3 >$ are two vectors and θ is the angle between them, then the dot product is the number

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = ab\cos\theta$$

2. properties of the dotproduct

3. orthogonal vectors iff $a \cdot b = 0$

4. direction angles (the angles of the vector with the axes) which can be found using

$$\cos\alpha = \frac{a_1}{|a|}, \cos\beta = \frac{a_2}{|a|}, \cos\gamma = \frac{a_3}{|a|}$$

5. $< \cos\alpha, \cos\beta, \cos\gamma >= \dfrac{a}{|a|}$

6. scalar projection of b onto a is $\text{comp}_a b = \dfrac{a \cdot b}{|a|}$

7. vector projection of b onto a is $\text{proj}_a b = \dfrac{a \cdot b}{|a|^2} a$

12.4 The cross product

1. if $a =< a_1, a_2, a_3 >$ and $b =< b_1, b_2, b_3 >$ are two vectors and θ is the angle between them, then the cross product is

$$a \times b =< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

2. $a \times b$ is orthogonal to both a and b

3. $a \times b = 0$ iff $a||b$

4. $|a \times b| = |a||b| \sin \theta$, where $0 \leq \theta \leq \pi$ which is the area of the parallelogram determined by a and b

5. properties of $a \times b$

6. volume of the parallelepiped determined by three vectors a, b, c is $V = |(a \times b) \cdot c|$ (since $a \times b$ gives the area of one side of the parallelepiped, and $c \cos \theta$ is the hight that is perpendicular to $a \times b$, obtaining $V = |(a \times b)| \, |c| \, |\cos \theta|$)

12.5 Equations of Lines and Planes

LINES:

1. Let P_0 be a point on a line L, and $\mathbf{v}$ be a vector parallel to L. A vector equation of a line L is $\mathbf{r} = \mathbf{r_0} + \mathbf{tv}$, where $\mathbf{v}$ is the vector parallel to L that gives the direction of the line L, $\mathbf{r}, \mathbf{r_0}$ are the position vectors of P and P_0, respectively (P is an arbitrary point on L).

2. parametric equations of line L (for $t \in \mathbb{R}$):

$$x = x_0 + a\,t,\ y = y_0 + b\,t,\ z = z_0 + c\,t$$

3. vector equation and parametric equations of a line are not unique

4. if $v =< a, b, c >$ is a direction vector for a line L, then a, b, c, are the direction numbers of L

5. symmetric equations $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ $(a, b, c \neq 0)$. If $a = 0$, then $x = x_0$ and L lies in the vertical plane $x = x_0$. Similarly for $y = y_0$ and $z = z_0$

6. the line segment from r_0 to r_1 is given by $\overrightarrow{P_0P} = \mathbf{r(t)} = (1 - t)\,r_0 + t\,r_1$ where $0 \leq t \leq 1$

7. two lines in space can be

   - parallel: if their corresponding vectors are parallel (i.e. components are proportional)
   - intersecting: if they share a common point (see if you can solve for parameters t and s in setting the corresponding parametric equations equal to each other)
   - skew: otherwise (i.e. you can't solve for s and t)

PLANES:

1. vector equation of the plane: $n \cdot (r - r_0) = 0$ or $n \cdot r = n \cdot r_0$ (it is a dot product because $n \perp (r - r_0)$)

2. scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $n =< a, b, c >$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

3. two planes can (1) be parallel (if their normal vectors are parallel), or (2) intersect in a line (angle between the planes is given by $\cos\theta = \dfrac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$) – a pair of linear equations represents the line of intersection of the planes.

4. a vector parallel to the line of intersection of two planes is given by $\mathbf{n_1} \times \mathbf{n_2}$

5. the distance D between a point P and a plane is given by

$$d = |comp_n b| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}},$$

   where $\mathbf{b}$ is the vector corresponding to P and a point P_0 in the plane

6. to find the x-intercept: set $y = z = 0$. Similarly for y- and z-intercepts

12.6 Cylinders and Quadratic Surfaces

1. cylinder: surface that consists of all the rulings (lines) that are parallel to a given line. If they are all parallel to one of the coordinate axis, then one variable is missing from the equation of the surface (i.e. $y^2 + z^2 = 1$ that has no x involved)

2. quadric surface: is a graph of a second degree equation in 3 variables (just like the conics are in the plane). Examples: ellipsoid (all of whose traces are ellipses), elliptic paraboloid (traces are ellipses and parabolas), hyperbolic paraboloid (traces are hyperbolas and parabolas), hyperboloids (of one or two sheets)

12.7 Cylindrical and Spherical Coordinates

1. cylindrical coordinate system: a point P in 3-dimensions is represented by the triple (r, θ, z), where r and θ are the polar coordinates, and z is the directed distance from xy plane to the point P

2. useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with the axis of symmetry (like cylinders, cones)

3. converting cylindrical to rectangular coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$

4. converting rectangular to cylindrical coordinates: $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$

5. spherical coordinate system: a point P in 3-dimensions is represented by the triple (ρ, θ, ϕ), where $\rho = |OP|$, θ is the angle, in radians, that OP maxes with the polar axis(positive x-axis), and ϕ is the angle between the positive z-axis and OP $(\rho \geq 0, 0 \leq \phi \leq \pi)$.

6. useful in problems where there is symmetry about a point, and the origin is placed at this point (like spheres, half planes, half cones)

7. converting spherical to rectangular coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$

8. converting rectangular to spherical coordinates: $\rho^2 = x^2 + y^2 + z^2$ (for exact points, like Example5 page 841, find θ and ρ using trig functions